THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H/I/J University Mathematics 2017-2018
Assignment 2
Due Date: 7 Feb 2018 (Wed)

1. State whether the following sequence converges. If no, just write "this sequence is not convergent". (There is no need to give reason.) If yes, find the limit.

   (a) $a_n = \dfrac{3^n - 1}{3^n + 1}$.

   (b) $a_n = (-1)^n$

   (c) $a_n = \sqrt{n+5} - \sqrt{n}$

   (d) $a_n = \cos \dfrac{n\pi}{2}$

   (e) $a_n = \dfrac{3n^2}{n+1} - 3n$

   (f) $a_n = (2 - \dfrac{1}{2^n})(3 + \dfrac{2}{n^2})$.

   (g) $a_n = (\sqrt[3]{n^2+1} - \sqrt[3]{n^2})$

   (h) $a_n = \left[\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\cdots\left(1 - \dfrac{1}{n^2}\right)\right]$

2. By using the sandwich theorem, evaluate the following limits.

   (a) $\lim\limits_{n\to\infty} \dfrac{6n + \cos n}{2n}$

   (b) $\lim\limits_{n\to\infty} \dfrac{2n^2 + (-1)^n n}{n^2}$

3. By using the sandwich theorem, prove that

$$\lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right) = 0.$$

   (Remark: You may not use the algebraic rules of limits since when n goes to infinity, we are summing up infinitely many terms. Hint: Try to consider $\dfrac{1}{(2n)^2} \leq \dfrac{1}{r^2} \leq \dfrac{1}{n^2}$ for all $n \leq r \leq 2n$.)

4. (a) Resolve $\dfrac{5x-3}{x(x+1)(x+3)}$ into partial fractions.

   (b) Hence, evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{5k-3}{k(k+1)(k+3)}$ (i.e $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{5k-3}{k(k+1)(k+3)}$).

5. Let $\{a_n\}$ be a sequence of real number defined by $a_1 = 0$ and $a_{n+1} = 2n - a_n$ for $n = 1, 2, 3, \cdots$.

   (a) By using mathematical induction, prove that for all integers $n \geq 1$,

$$2a_n = 2n - 1 + (-1)^n.$$

   (b) Hence, evaluate $\lim\limits_{n\to\infty} \dfrac{a_n}{n}$.

6. (a) Prove that $\dfrac{2^n}{n!} \leq \dfrac{4}{n}$ for all natural numbers $n \geq 2$.

   (b) Hence, show that $\lim\limits_{n\to\infty} \dfrac{2^n}{n!} = 0$.

7. Let $\{x_n\}$ be a sequence of positive real numbers defined by $x_1 = 2$ and $x_{n+1} = x_n^2 - x_n + 1$ for all positive integers n. Define $s_n = \displaystyle\sum_{i=1}^{n} \frac{1}{x_i}$ for all positive integers n.

(a) Prove that for any positive integer n,

(i) $x_n > n$,

(ii) $s_n = 1 - \dfrac{1}{x_{n+1} - 1}$.

(b) Hence, prove that $\lim_{n \to \infty} s_n$ exists.

8. Let $\{x_n\}$ and $\{y_n\}$ be sequences of positive real numbers such that $0 < y_1 \leq x_1$ and

$$x_{n+1} = \frac{x_n + y_n}{2} \qquad \text{and} \qquad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

for $n = 1, 2, 3, \cdots$.

(a) Show that $x_n \geq y_n$ for all natural numbers n.

(b) Prove that $\{x_n\}$ is a monotonic decreasing sequence and $\{y_n\}$ is a monotonic increasing sequence.

(c) Prove that $\{x_n\}$ and $\{y_n\}$ converge and $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n$.

(d) Prove that $x_n y_n$ is a constant and hence find $\lim_{n \to \infty} x_n$ in terms of x_1 and y_1.